The Research on Graduate Students' Understanding of Three Basic Limit Concepts

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(Corresponding Author)

Wang Caifen[1]
Fu Hailun[2]
Cao Rongrong[3]

[1]*School of Mathematics and Statistics, Shandong Normal University, Jinan, Shandong, China; School of Mathematics and Statistics, Qingdao University, Qingdao, Shandong, China*
[2]*School of Mathematics and Statistics, Shandong Normal University, Jinan, Shandong, China*
[3]*School of Mathematics and Statistics, Qingdao University, Qingdao, Shandong, China*

ABSTRACT

This research adopted the methods of semi-structured interviews and questionnaires, chose 40 first year graduate students as participants, studied what opinions they held about the relationship between the three limits which are limit of a sequence, limit of a function at a point and limit of a function at infinity. The study shows that most of students' understanding of the concept of limits is incomplete. They believe that three limit concepts are distinct but related, and cannot be integrated into a limit concept. Misconceptions and cognition obstacle of the limit exist extensively in the mind of graduate student, and it is very difficult to overcome for them. It is necessary for both teachers and textbooks to clarify the "common" of the three limits and provide a unifying organizational framework to all three concepts of limit for students. These new findings expand and deepen the current research about students' understanding of three limits.

Keywords: *Limit of a sequence, Limit of a function, Cognition obstacle.*

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1. INTRODUCTION

The concept of limit is one of the most fundamental ideas not only in understanding calculus but also in developing mathematical thinking beyond calculus and in pursuing mathematical rigor (Tall, 1992). Many mathematical concepts are based on the concept of limit: the derivative, the integral, continuity and the sum of an infinite series (Roh, 2008).

Students are introduced to three basic notions of limits: limit of a sequence $\lim_{n\to\infty} a_n$, limit of a function at a point $\lim_{x\to x_0} f(x)$, and limit of a function at infinity $\lim_{x\to\infty} f(x)$, so are in many textbooks whether elementary calculus or mathematical analysis (Department of Mathematics of Tongji University, 2014).

The formal definitions of the three limits are similar in terms of the mathematical symbol used, the structure and their meaning. Their connection and unification are rarely given in many textbooks. An underlying conceptual framework was also rarely made explicit (Department of Mathematics of East China Normal University, 2011).

In general, most teachers think that students could naturally encapsulate the three limits in one concept with the notion of 'neighborhood' : $\lim_{X\to A} f(X) = L$ if for every neighborhood V of L there is a neighborhood U of A with $f(U\setminus\{A\})\subset V$ (Fernández-Plaza and Simpson, 2016). Then students would possess a unifying organizational framework for all three limits. The research will consider the following questions

1) How do students understand the relationships of the three concepts?

2) Do students regard them as a unified notion of limit, as distinct but related or as separate notions?

3) What are the reasons that students hold such opinions?

2. LITERATURE REVIEW

Cornu (1991) thinks that the difficulty in teaching and learning lies not only in the richness and complexity of the limit concept itself, but also in that the limit definition itself is not enough to generate the cognitive aspects needed to understand the concept. It is one thing to remember the definition of the limit, and to grasp the limit concept is another [p153]. Tall (1992) argues that Limit is the first mathematical concept that students meet where one does not find the result by a straightforward mathematical computation.

There are many factors that influence teaching and learning of limit concept, but, according to Cornu (1991) the following four factors are the most important ones.

1) Concept definition and concept image. Influenced by the everyday Words (" tend to be ", "limit") or daily experience (Schwarzenberger and Tall, 1978) so many concept images are developed in the mind of the students which are different from the formal concept definition and contain factors which cause cognitive conflict, such as "getting close", "growing large" and so on (Tall and Vinner, 1981; Cornu, 1991). These cognitive conflicts will hinder students' further understanding of limit concepts.

2) Psychological obstacles which occur as a result of the personal development of the student. According to the Process-Objects Duality Theory (Sfard, 1991) many mathematical concepts represent not only operational process but also a mathematical object or structure. Students often experience the process from the beginning of the process to the transformation of the object when they develop a mathematical concept. The two co-exist in the cognitive structure ultimately and work together at the appropriate time (Li and Wu, 2011). Vinner (1991) argued that

students often regard the infinite process itself as the limit rather than seeing the limit as the result of the infinite process. It was very hard to internalize and encapsulate process into an object for most students.

3) Didactical obstacles which occur because of the nature of the teaching and the teacher (Cornu, 1991). The teachers' teaching deeply affects the students' understanding of the concepts of limit (Roh, 2008; Fernández-Plaza and Simpson, 2016) because some teachers cannot provide high level classroom instruction due to the limitation of pedagogical knowledge (Cao, 2011). For example, the teacher cannot give students the correct cognitive roots nor to design appropriate teaching situation.

4) Epistemological obstacles which occur because of the nature of the mathematical concepts themselves. It can be found both in the historical development of scientific thought and in educational practices. In formal definition of limit, " for any small quantity" implies the concept of "infinitesimal", "N growing large " implies the concept of "infinity" (Li and Wu, 2011). In the history of mathematics, mathematicians and philosophers seem to have never stopped the discussion and controversy on infinite thinking (" potential infinity", "actual infinity" and "infinitesimal" et.al.) (Kleiner, 2001; Tall and Tirosh, 2001).

These misunderstanding about limits and cognitive obstacles are unavoidable (Davis and Vinner, 1986) and are difficult to overcome (Williams, 1991; Cao, 2011) even exist at a more advanced stage of their studies. It doesn't seem to influence them to work out exercises, solving problems and succeeding in their examinations (Cornu, 1991).

Research on understanding limits has a long history, but there are few papers in the rich literature on students' understanding the connection of three notions of limit.

Fernández-Plaza and Simpson (2016) focused on students' understanding about three notions of limits which involved undergraduate mathematics students in second year of college, in particular, whether they seen them as manifestation of a unified limit concept, distinct concept with links between them or as disjoint concepts. The study shows that some students treat limit cases as separate, some can draw connections, but often do so in ways which are at odds with the formal mathematics, no student has a unifying organizational framework for three limits. On the basis of Fernández-Plaza and Simpson (2016) research, we continue to investigate the understanding of graduate students about the link of the three limit concepts.

3. METHOD

3.1. Participants

40 graduate students were chosen. They are in their first year studies of master degree. 15 of them are majoring in computational mathematics, 13 students are majoring in applied mathematics, other 12 are statistics. All graduate students have undergone rigorous academic training in mathematics major for 4 years. They all passed the rigorous postgraduate entrance exam which was sponsored by Chinese education government.

3.2. Materials

There are two parts of the questionnaire. One is to examine students' opinion on relationship between three limits:

1) They are separate and unconnected concepts

2) They are distinct but related concepts.

3) They are a unified concept.

Another is to explore whether students can unify the three concepts within an organizational framework.

3.3. Data Collection and Analysis

Students were first asked to do a written questionnaire. There was no time limit on the written questionnaire it was done within the whole morning. The questionnaires were completed independently and discussions were not allowed, and then used as basis for the subsequent interviews. 7 students were chosen to have semi-structured interviews. Each of them was conducted one-to-one by the first author. The participants were allowed as much time as they wished to complete the interview, generally taking between 30 min and 1 hour. The interviews were recorded and transcribed. The results were analyzed by the authors in group.

4. RESULTS

4.1. Students' Opinion on Relationship Between Three Limits

Our analysis of the data suggests that there are five main types of making sense of the three limit concepts for students, as shown in table 1 (For convenience, the limit 1, 2, 3 represent Limit of a sequence , limit of a function at infinity, Limit of a function at a point).

Type1: They are separate and unconnected concepts

Type2: Limit 1 and 2 has connections. But they have no relationships with limit 3.

Type3: Limit 2 and 3 has connections. But they have no relationships with limit 1.

Type4: Limit 1and 2, limit 2 and 3 have connections respectively. But limit 1 and 3 have no relationships.

Type5: Others.

Table-1. Five main types of making sense of the three limit concepts.

Type1		**Type2**		**Type3**		**Type4**		**Type5**	
No.	P.c.	No.	P.c.	No.	P.c.	No.	P.c.	No.	P.c
5	12.5	9	22.5	13	32.5	11	27.5	2	5

Source: Research findings, 2017

4.1.1. Type 1: They are Separate and Unconnected Concepts

5 students (around 12.5%) held this view. They gave reasons such as "I don't remember our teacher tell us any relationships between them", "In the textbook, I didn't find any relevant content about relationships between them" or "I have no idea about this".

4.1.2. Type 2: Limit 1 and 2 has Connections. But they have No Relationships With Limit 3

There were 9(around 22.5%) students holding this opinion. They argued that the independent variable of limit of a sequence tends to infinity， so does limit of a function at a point. Limit of a sequence can be regarded as limit of subsequence of a function. According to "the values (infinity and finite value) of the independent variables tend to", three limits can be divided into two groups: one group was limit of function at infinity (including limit of a sequence and limit of a function at infinity), another was limit of function at a point. There was no relationship between two groups.

Researcher(R): Please talk about how you thought about the question according to your written questionnaire?

Student 1(S1): a sequence can be seen as a subsequence of function that the independent variables are positive integers, for an example, $\lim_{x\to\infty}\frac{1}{x}$ and $\lim_{n\to\infty}\frac{1}{n}$, both independent variables tend to infinity, So limit of sequence can be regarded as a special limit of a function. So…the two limits are the limits at infinity. So…So three limits can be grouped two categories: limit at a point and limit at infinity.

S2: When $x \to \infty$ or $n \to \infty$, Infinity is at infinity and can never be reached, limit also can't reach. So limit 1 and 2 have connection .When $x \to x_0$, limit can reach, so limit 3 is different from limit 1 and 2.

4.1.3. Type 3: Limit 2 and 3 has Connections. But they have no Relationships with Limit

There were 13(around 32.5%) students holding this opinion. They believed that the graph of function was continuous, but the sequence wasn't. There was connection between limit of function at a point and limit of functions at infinity since their graphs were continuous. Both limits had no connections with limit of a sequence since the graph of a sequence was not continuous.

R: Why did you think them according to "whether the graph of function is continuous or not"? Could you explain how did you think about it at that time?

S3: Infinity is a very large number, x_0 also is a number. So $x \to x_0$ and $x \to \infty$ are the same in nature. so…um…so two limits $\lim_{x \to x_0} f(x)$ and $\lim_{x \to \infty} f(x)$ have connections. Because… the graph of limit of sequence is discrete…but…the functions' graph is continuous. So …so according to the continuity ,these three limit concepts can be divided into two types, one is sequence limit, another is function limit. There is no relationship between two groups.

4.1.4. Type 4: Limit 1and 2, Limit 2 and 3 have Connections Respectively. But Limit 1 and 3 have No Relationships

There were 11(around 27.5%) students holding this opinion. Some students did not give the reason. One student (whose code is S4) gave reasons below.

S4: Three limits are distinct but related. Look at the examples, $1) \lim_{n \to \infty} 2^{-n}; 2) \lim_{x \to \infty} 2^{-x}; 3) \lim_{x \to 0} 2^{-x}$. 2^{-n} is a subsequence of 2^{-x}. Their Independent variables tend to infinity, and they have the same limit value. So they have connection. Limit 2 and 3 have the same functions, their limits also exit even though different numbers, so they have connections too. Other students took agree with this student.

4.1.5. Type 5: Others

There are 5% （only 2） students whose opinions we could not identify.

In summary, more than 80% of students believed that the three limits were different but related. Only 12.5% of students believed that they were independent and unrelated. No one student said that they were a unified concept.

4.2. Whether Three Limits are a Unified Concept or Not

When students were asked whether three concepts could be integrated into one concept or not, 70% students gave answers "no" (as shown in table 2). 20% students believed that It seemed that three limits can be integrated into one concept, but it was difficult or impossible to give a unified definition. Only 2 (around 5%) students could unify three concepts within an organizational framework. One suggested that three limits were the same in nature, and could be integrated into a unified definition which was $\varepsilon - \delta$ definition of limit of function at a point. Then he explained it below.

S5: Sequence can be regarded as the function which independent variable is the positive integer, um…so you can see the limit of sequence as …a special case of the limit of the function at infinity… and …then make a transformation…um…in the expression $\lim_{x\to\infty} f(x)$, let $y = \frac{1}{x}$ …then $x \to \infty$ means $y \to 0$, the expression $\lim_{x\to\infty} f(x)$ becomes $\lim_{y\to 0} f(y)$, the limit of this function at infinity becomes limit of function at a point. Um…Then… can be unified to the limit of function $\lim_{x\to x_0} f(x)$ …It has ε-δ neighborhood.

Another student also expressed similar views.

Table-2. A unified concept

No		seems		Yes	
No.	P.c.	No.	P.c.	No.	P.c.
30	75	8	20	2	5

Source: Research findings, 2017

5. CONCLUSION AND SUGGESTION

*Conclusion 1:*Understanding to three basic limit concepts of overwhelming majority of students is one-side since that 80% students believe three limit concepts are distinct but related concepts and 75% students argue that they cannot be integrated into a unifying limit concept. This also means vast majority of students can't unify the three concepts within an organizational framework.

Mathematics education research shows that the construction of mathematical knowledge need to go through two stages--- operation and reflection abstract (Sfard, 1991; Li and Wu, 2011). Compared to operation stage, reflection abstract is more complicated and difficult to control which need more guidance and help from the teacher. some students hold that "they had considered that the three limits were similar when they began to learn the concept of limit, it seemed to have a unified concept for three limits, but did not think deeply about the question in that time, then soon over, and never reflect about this question later", or "I took into account the relationships between three limits, but I did not work out, then never think about this matter carefully". Obviously, these students had begun to reflect on the three concept of limit, but it was hard for them to work out for the answer. Or due to lack of time to reflect carefully, or the instructor didn't provide students an appropriate opportunity to help students to integrate three limits, the cognition of these student to the limit has not been improved much. It is not easy to successfully develop a correct understanding of the concept of limits.

Conclusion 2 : Misconceptions and cognition obstacle of the limit existed extensively in the mind of graduate student, and it was very difficult to overcome for students. The research object of this paper is the first-year postgraduates majoring in mathematics. After four years of professional mathematics training, they also passed the national postgraduate entrance examination initiated by the Ministry of Education of China. However, in the interview, some students emphasized that "whether limit process can be reached or not", which shows that the students' understanding of the limit concept is still in the process stage, and the process of limit concept was not internalized into the object of limit. "Infinity is at infinity and can never be reached", which shows that students' understanding of infinity was still in the phase of potential infinity, and was not up to the actual infinity. Students' cognitive obstacle about infinity concept seriously affected the students' understanding of the concepts of limit.

The finding above supports the work of Davis and Vinner (1986); Williams (1991) and Cornu (1991) such as "unavoidable", "It is difficult to remediate", "even exist at more advanced stage of their studies".

At the same time, the researchers also found that misunderstanding of the concept of limit do not seem to prevent them from working out exercises and succeeding in their exams, because they are all excellent students who succeeded in National Postgraduate Entrance Examination in China. This finding is in line with the work of Cornu (1991) and others. "Remembering the concept is one thing, but understanding concept is another".

Based on the above research results, we give the following teaching suggestions.

In classroom teaching, it is necessary for teachers to help students to integrate the concepts of different limits through various teaching methods. As Fernández-Plaza and Simpson (2016) suggested, when the limit of the function is introduced, teacher need to make connections between limit of sequence and limit of function more explicit, and guide students to pay attention to the common or unique features of different types of limits, and help students to relate these concepts to each other; It will also be good teaching method to define the limit concept in one context, and to make sense of it in another as Swinyard (2011) mentioned.

As the teaching material that students used, it is very important to interpret and reveal clearly commonness of the three types of the limits, and provide students a unifying organizational framework for three limits to help students not only understanding of connection of three concepts of limit, but also help them to remediate the incorrect cognition of limit.

In addition, the teachers need to consider the various cognitive obstacles of the limit concept and its corresponding teaching strategies in the practice of college mathematics teaching. There were heated debates over the questions, as follows, "Is the limit attained or not", "Does infinitesimal exist or not?", "potential infinity" and "actual infinity" and other problems in the history of the concept of limit. The researchers argue that it shouldn't be avoided just because they are too difficult. Instead, it is necessary to provide the learning environment in which students are made aware of difficulties and to give the opportunity to reflect on their own ideas and cognitive obstacles (Fischbein, 1987; Cornu, 1991; Brousseau, 2006). It is also an important part of learning of mathematical concepts for students to remediate the incorrect cognition of limit.

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