# Enhancing mathematical modelling competency through textbook design

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## ABSTRACT

Around the world, mathematics school curricula increasingly use learning goals in an effort to capture different kinds of processes that students should master. For primary and secondary mathematics education in Denmark, these ambitions are expressed in terms of a set of mathematical competencies. However, implementing such competencies into actual teaching practice has proved challenging particularly when it comes to mathematical modelling competency. Matematrix is a Danish mathematics textbook system for grades K-9, designed to support mathematics teachers in overcoming this challenge. In this article, I as one of the designers and authors of these textbooks describe and exemplify their design and the analytical foundations on three different levels: the overall structuring of the books' content, the focal points for each chapter, and the development of different kinds of tasks for students to work with. In these descriptions, I focus on and provide concrete examples of how my colleagues and I sought to support teachers in facilitating their students' development of mathematical modelling competency. This analytical focus leads me to distinguish between and exemplify mathematization tasks of short duration, constructive modeling tasks of short duration, and inquiry-based projects of longer duration focusing on either the constructive or the receptive facets of mathematical modelling competency.

**Keywords:** *Application-reflective competency, constructive mathematical modelling competency, mathematisation competency, mathematisation tasks versus modelling tasks, modelling tasks versus inquiry-based modelling projects, textbook design, two-dimensional content model.*

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**Highlights of this paper**

- Use a two-dimensional description of content.
- Distinguish between constructive mathematical modelling competency, application-reflective competency, and mathematization competency.
- Provide invitations for both inquiry-based modelling projects and desk-based modelling work of short duration.

## 1. INTRODUCTION

Over the past 40 years, mathematics curricula worldwide have undergone reforms characterized by a shift from subject matter learning goals related to mathematical concepts and procedures to learning goals that combine mathematical subject matter with mastery-oriented ambitions (Højgaard & Sølberg, 2023). Numerous reports and books describe this shift in various national contexts, for example (Niss, Bruder, Planas, Turner, & Villa-Ochoa, 2016). Australia (Australian Education Council, 1994), Denmark (Niss & Jensen, 2002), Korea (Lew, Cho, Koh, Koh, & Paek, 2012), Latin America (Tobón, Pimienta, & García, 2010), Germany (W. Blum, Drüke-Noe, Hartung, & Köller, 2006), Portugal (Abrantes, 2001), and the US (National Research Council, 2001; NGA Center & CCSSO, 2010).

In Denmark, the publication of the KOM report (Niss & Jensen, 2002) initiated a developmental approach, where mastery-oriented curricular ambitions are described by means of a set of mathematical competencies. In 2009, the set of mathematical competencies proposed in the KOM report became a fundamental part of the curriculum for compulsory mathematics education (grades K-9) in Denmark (Undervisningsministeriet, 2009), which remains the case following the most recent curricular reforms (Børne- og Undervisningsministeriet, 2019). This process of curricular development is thoroughly described and analyzed in Højgaard and Sølberg (2023).

One of the main findings of research studying reforms that focus on student mastery is that they often do not have the desired impact on classroom teaching practices (Cuban, 2013; Hopmann, 2003), resulting in a gap between the intended and implemented curriculum (Bauersfeld, 1980). Denmark has not been an exception, with significant efforts made to implement the formal curriculum's good intentions for competency-oriented mathematics instruction in classrooms. I have been involved in this developmental process in various ways, but I will focus on one particular aspect of this involvement in this article: From 1998 to 2020, I was one of the leading figures in the development, writing (1998–2008), and subsequent revision (2010–2020) of a series of mathematics textbooks, *Matematrix*, for use in primary and lower secondary education in Denmark (grades K–9). The three years prior to the publication of the first book in 2001 were spent deciding on the books' fundamental approach and an appropriate didactic design. This process took place parallel to my involvement in the KOM project. Hence, one of the main ambitions in developing the new textbook system was to systematically facilitate the incorporation of mathematical competencies as a key element of mathematics education for grades K–9 in Denmark. When the process of revising the textbooks began in 2010, the KOM competencies had just been placed at the core of the curriculum, as outlined above. It was therefore only natural that one of the key goals of the revision was to be even more systematic, thorough, and explicit in facilitating students' development of mathematical competencies.

In this article, I focus on one aspect of this process, exploring the following research question:

How can textbooks be designed to systematically facilitate students' development of mathematical modelling competency, and what types of tasks best support this development?

In answering this question, my analysis is framed by an approach and model used to characterize content and objectives in the textbooks' various parts. I begin by presenting this model, cf. the detailed analysis in Højgaard (2025a). Naturally, the analysis is also based on a particular conceptualization of mathematical modeling competency, which I will introduce, as well as highlighting three related sub-competencies that I consider important to address in teaching. During this analysis, I describe and exemplify how we chose to support students' development of each of these three sub-competencies in the Matematrix books, both at a structural level, following the general model for content and objectives, and at the level of task design. In all cases, I have translated passages into English from the original Danish. The article concludes with a summary of my analytical findings regarding the research question, as well as a discussion of the relevance of this kind of research.

## 2. METHODOLOGICAL APPROACH

The chosen methodological approach is straightforward: The Matematrix textbooks are *a data source*, as are the accompanying teachers' manuals with explicit descriptions of some of the underlying considerations. To "peek behind the curtain" and elaborate on the rationale for the books' final form and content, I make use of unpublished schematics and models from the developmental process. More importantly, I rely on what Lincoln and Guba (1985) call *human instruments*; that is, the researcher "elects to use him- or herself as well as other humans as the primary data-gathering instruments" (p. 39). In this case, I am that instrument, since I was not only present when we myself and various editors and co-authors analyzed, discussed, and decided on the approach to developing and writing the books, but also the driving force behind these structural decisions. The same was true of the decisions about and writing of tasks to support students' development of mathematical modeling competency, as that was the main research and development focus I brought into the Matematrix group (Jensen, 2007).

In this article, the development and final design of the Matematrix textbooks are used as *a case study* of the type described by Merriam (1998) as "an intensive, holistic description and analysis of a bounded phenomenon" (p. xiii).

## 3. THE RELATIONSHIP BETWEEN COMPETENCIES AND SUBJECT MATTER

When Mogens Niss, the chairman of the KOM committee, and I accepted the Danish Ministry of Education's invitation to be the driving forces in the KOM project around the turn of the millennium, our main motivation was to fight *syllabusism* in Danish mathematics education. As discussed in Højgaard (2024), syllabusism denotes the equation of mastery of a subject with proficiency in specific subject matter, which is placed at the core of all educational processes, from teaching to curriculum development.

### *3.1. A Two-Dimensional Description of Content*

This motivation led us and the rest of the KOM working group to focus on the relation between competencies and subject matter areas in the curriculum, and to suggest they be considered two independent dimensions of curricular content, represented in a matrix structure as depicted in Table 1 (Niss & Højgaard, 2019; Niss & Jensen, 2002). Højgaard and Sølberg (2023) offer an analysis of the lessons to be learned from the developmental process of translating this idea, which lies at the core of the KOM project, to mathematics curricula for primary and lower secondary education in Denmark.

**Table 1.** A matrix structuring of competencies × subject matter areas within mathematics education

| Subject matter area Mathematical competency | Area 1 | Area 2 | … | Area *n* |
|---|---|---|---|---|
| Mathematical thinking | | | | |
| Problem handling | | | | |
| Modelling | | | | |
| Reasoning | | | | |
| Representation | | | | |
| Symbols and formalism | | | | |
| Communication | | | | |
| Aids and tools | | | | |

**Source:** Niss and Jensen (2002) and Niss and Højgaard (2019)

### *3.2. A Competency Matrix for Each Book*

Following the approach developed in the KOM project, a pivotal part of the endeavor to systematically facilitate the incorporation of mathematical competencies as a key element in mathematics education is to separate mathematical competencies and subject matter areas as two independent dimensions of content (Højgaard, 2024). Subsequent research and development work has supported the importance of such an approach (Højgaard, 2019, 2021; Højgaard & Sølberg, 2023).

Hence, it was decided that the content and underlying learning objectives in the Matematrix textbooks for grades K-9 would comprise two independent dimensions: on one hand, a set of mathematical competencies, and on the other hand, a more concrete version of subject matter areas, a core syllabus consisting of a set of 27 fundamental mathematical concepts derived from the curriculum at the time (Undervisningsministeriet, 1995). Højgaard (2025a) presents an analysis of this structural framing of the development of the Matematrix books. In short, the conceptual analysis of the curriculum was used to decide on the conceptual progression through grades K-9 and, following this general analysis, on the conceptual focus of each book. These choices were combined with a set of mathematical competencies to form a so-called *competency matrix* for each book: competencies × chapter headings. As an example, Table 2 shows the competency matrix for *Matematrix 6*, provided in the accompanying teachers' manual (Gregersen, Jensen, Petersen, & Thorbjørnsen, 2020).

**Table 2**. An example of the competency matrix – competency objectives × chapter headings – for grade 6. Similar matrices accompany each book in the mathematics textbook series Matematrix.

| **Chapter Algebra Competency** | | **Movements** | **Equations** | **Drawing** | **Fractions** | **Percentages** | **Relationships** | **Statistics and probability** | **Formulas** | **Reality and mathematics** |
|---|---|---|---|---|---|---|---|---|---|---|
| Modelling | | Patterns as geographical models. Using movements as a tool for constructing patterns. | | Drawings as geometric models, including a choice among three drawing models. | | | Tables, graphs, equations, etc., serve as models of relationships from reality. | | Construction of formulas as models of various relationships. | Arithmetic tasks, formulas, and drawings as modeling tools. |
| Symbol handling | Letters as place holders for unknown numbers. Preparing for the concept variable. | | Decoding, construction and transformation of equations. | | A fraction is understood as a relation between numbers and as a representation of a number. | Decoding and calculations with percentages, fractions, and decimal numbers. | Linguistic description of various symbolically given relationships. | | Decoding existing formulas. Construction of one's own formulas. | Construction and use of one's own arithmetic tasks and formulas. |
| Representation | Unknown numbers represented by letters. | | Unknown numbers represented by letters in equations. | Various drawings as different representations of the same object. | Transformation from for example fraction to decimal number. | Percentages as representations of scales. | Math relationships are represented by, for example, everyday language, equations, and tables. | | Formulas as a way to represent relationships. | Choosing between arithmetic tasks, formulas, and drawings as modes of representation. |
| Communication | | Comm. about the elements in patterns, and about how geo. fig. can be moved. | | | | The communicative power of percentages, e.g., when comparing fractions. | Comm. about various relationships between unknown quantities. | Comm. about the use of statistics and probability. | Comm. about the relationships between unknown quantities | Communication about models made by oneself and others, and interpretation of their results. |

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | given by formulas. | |
| Aids and tools | | Using pairs of compasses, coordinate systems, protractors, mirrors, etc. | | Using isometric paper and exact tools to draw long, straight lines. | The calculator as a tool for transforming fractions to decimal numbers. | The calculator as a tool for transforming scales to percentages. | | | Decoding and constructing formulas in spread-sheets. | |
| Problem handling | | | *Solve the equation* $x + 1 = x$. | | Can you calculate with all fractions? | What is the smallest number you can get when finding percentages of something? | | What is the mean of five subsequent integers? | | |
| Reasoning | Reasoning about the validity of arithm. rules, e.g. opposite numb. and operations. | Reasoning about patterns and symmetry as geometric properties. | Relating to statements about eq.- and their solution (testing possible solutions). | | | Deciding if statements about calculations of percentages are true or false. | | Reflections on the quality of arguments based on statistics and probability. | Are these formulas the same? $E = 2 \cdot L$ and $L + L = E$ | |
| Application-critical | | | | Critically considering the choices a designer/painter has to make. | | | Experiments as a means to examining hypotheses about relationships. | Assessing the fairness of arguments based on statistics and probability. | Critically relating to other people's construction of formulas. | Critically relating to other people's models and interpretations of results. |
| Structural | On opposite numbers and opposite arithmetic operations. | Symmetry as a geometric property. | | | Different fractions as representations of the same rational number. | | | | | |
| Culture-historical | | | The history of the equation. | | | | Time zones as a way to define time different places on the Earth. | | | Currencies and rates of exchange in other countries. |

**Source:** Gregersen et al. (2020) and Højgaard (2025a)

### *3.3. Using the Competency Matrices at a Chapter Level*

The competency matrices have been used as developmental tools at both the chapter and task design levels (Højgaard, 2025a). Some of the chapters were primarily decided on and developed with a specific competency in mind; for example, the chapter *Reality and Mathematics* in *Matematrix 6* is devoted to mathematical modeling, and the explanations at the core of the chapter (Matematrix *6*, pp. 132-133) concern the mathematical modeling process rather than specific mathematical concepts. This approach continues in the subsequent books in the series, concluding with Matematrix 9, which explicitly introduces the concepts "mathematical model" and "mathematical modeling" (Matematrix 9, pp. 12-13).

### *3.4. Using the Competency Matrices at a Task Design Level*

For task design purposes, the competency matrices were used to decide on and communicate 3-4 competencies as the explicit focus and proposed learning objectives for each chapter, cf. the yellow cells in Table 2. To make these decisions more concrete and binding, a list of proposed learning objectives is provided for each chapter in the accompanying teachers' manual. For example, the teachers' manual for the chapter on *Relationships* in Matematrix 6 (Matematrix 6, teachers' manual, p. 110) outlines:

- 2-4 learning objectives with a conceptual focus, e.g., "[…] the student develops an understanding of what it means that there is a relationship between different things", stemming from the concepts × grades model, and
- 3-4 competency objectives, e.g., "[…] the student can represent mathematical relationships in different ways and gain experiences with the strengths and weaknesses of different representations", stemming from the competencies × chapter headings model.

As a design principle, each of the stated competency objectives is accompanied by a list of tasks (from the chapter in focus) explicitly designed with that objective in mind, and they are also addressed in the comments on each task in the teachers' manual. For example, the competency objective outlined above regarding strengths and weaknesses of different representational modes was used to generate the following task (Matematrix *6*, p. 134): "Use different mathematical tools to show a-e. […] c. How can one draw a sunset over the sea?"

## 4. MATHEMATICAL MODELLING COMPETENCY

The teaching and learning of mathematical modelling has been on the research agenda for mathematics education for decades (Blum & Niss, 1991; Kaiser & Brand, 2015). Two positions seem to dominate, distinguished by the role attributed to mathematical modelling (Højgaard, 2021).

The first position considers mathematical modelling *a means* – a didactic tool – to achieve other learning goals. Blum and Niss (1991) refer to this as *the promoting mathematics learning argument*. An example of such a perspective is the concept of model-eliciting activities developed by Richard Lesh and colleagues (Lesh & Doerr, 2003).

The second position, *the formative argument* (Blum & Niss, 1991)**,** regards the ability to carry out mathematical modelling *a goal* in itself for mathematics education. From this position, it is an important analytical task to describe and discuss the foci of this goal, and how to facilitate and support students in achieving it. In the remainder of this article, I carry out such an analysis in relation to mathematical modelling competency and its facilitation in the Matematrix textbooks.

### *4.1. Characterizing the Core Features of the Competency*

I use the term *competence* to denote a person's insightful readiness to act in response to the challenges of a given situation (Blomhøj & Jensen, 2003). Consequently, *a mathematical competency* is a person's insightful readiness to act in response to a certain kind of mathematical challenge in a given situation.

Following this conceptualization, I use *mathematical modelling competency* to describe an individual's insightful readiness to carry out all steps in a mathematical modelling process within a specific context and to critically analyze mathematical models produced by others (Blomhøj & Jensen, 2003; Jensen, 2007).

### *4.2. The Mathematical Modelling Process*

I understand "a mathematical modelling process" as denoting a complex and often not especially streamlined process involving a wide range of ways of thinking and acting. I have often found it helpful to work with a model that describes this process in six phases, as shown in Figure 1.

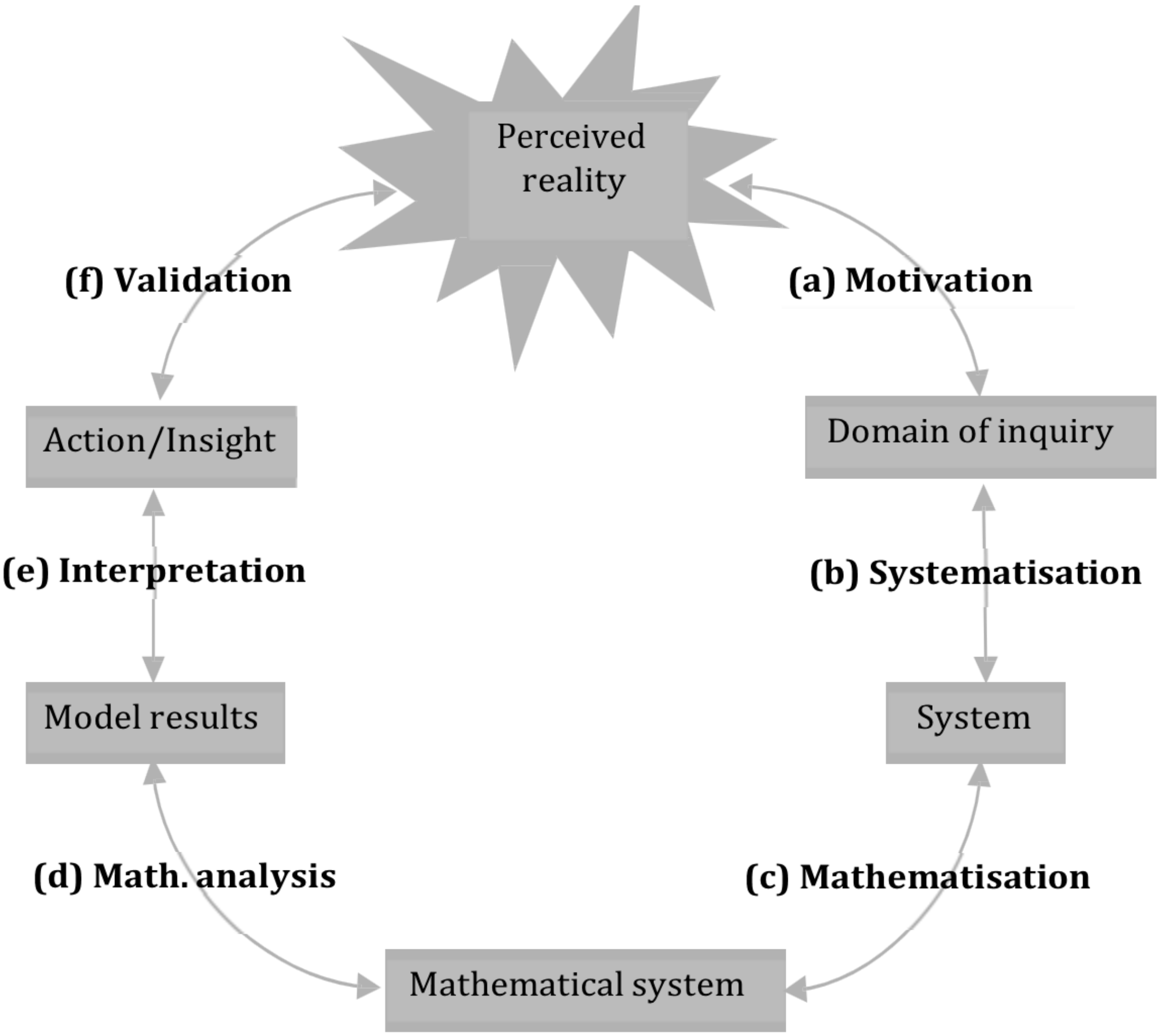


**Figure 1**. A visual model of the mathematical modelling process.

**Source:** Blomhøj and Jensen (2007).

A more detailed description of these six phases can be found elsewhere (Blomhøj & Jensen, 2003; Blomhøj & Jensen, 2007; Højgaard, 2021). The model is inspired by many similar models of the mathematical modeling process that can be found in the research literature on mathematics education (e.g., Blum & Leiß, 2007; Niss, 2010; Niss & Blum, 2020).

### *4.3. Textbook Approach: Focus on Three Modelling Sub-Competencies*

A common feature of the various models of the modelling process is that they intrinsically favor descriptions of different sub-abilities that comprise mathematical modelling competency (Kaiser & Brand, 2015). When analyzing and deciding on the approach to mathematical modelling competency in the Matematrix books, we used this feature of the six-phase model in Figure 3 to identify and address three modelling sub-competencies. In the following, I address these three sub-competencies in turn by outlining their core characteristics and describing how they are facilitated in the textbooks. The latter is done by means of translated exemplary tasks and links to the relevant pages (in Danish) from the revised and updated second editions of the various Matematrix books.

Two of the three modelling sub-competencies arise from the fact that, like its "siblings" in the KOM report, the conceptualisation of mathematical modelling competency provided above reflects that possessing it involves a duality between two sides of the same coin: *a receptive facet* ("critically analyse mathematical models produced by others"), focusing on understanding and critically assessing existing processes, and *a constructive facet* ("carry out all steps in a mathematical modelling process"), focusing on the ability to conduct such processes oneself, in this case, mathematical modelling (Niss & Højgaard, 2019; Niss & Jensen, 2002).

Analytically, this duality, stemming from the KOM framework, makes perfect sense, but from a teacher's perspective, it is not very helpful. When planning competency-oriented mathematics teaching, it is difficult to combine a constructive approach that invites students to conduct mathematical modelling themselves with an approach encouraging them to critically assess existing modelling processes, as I exemplify below. The former requires open-ended, extra-mathematical tasks for students to model, whereas the latter requires that students establish or be invited into a situation where they believe someone else has conducted mathematical modelling that they can relate to and critique.

To consistently facilitate the teaching of both facets of mathematical modelling competency and to clearly communicate the differences between them to teachers using Matematrix, we therefore labelled and worked with the two facets of modelling competency as two distinct sub-competencies: constructive mathematical modelling competency and application-reflective competency.

## 5. CONSTRUCTIVE MATHEMATICAL MODELLING COMPETENCY

### ***5.1. Characterizing the Core Features of the Sub-Competency***

As the name indicates, I use *constructive mathematical modelling competency* to describe the constructive facet of mathematical modelling competency that is, someone's insightful readiness to carry out all steps in a mathematical modelling process within a specific context, cf. the conceptualisation of mathematical modelling competency above.

As elaborated in Højgaard (2021), this inevitably entails a working process characterized by the need for various forms of demarcation and clarification in order to enable mathematical representation and analysis of an extra-mathematical challenge. These stages correspond with the phases labeled *motivation* and *systematization* in Figure 3. Due to the "underdetermined" nature of these initial stages of the mathematical modeling process, the core challenge is to learn how to manage the many choices that need to be made before mathematical concepts and techniques can be applied.

Seen through a didactic "competency lens," constructive mathematical modelling is mainly of interest if it provides a way of coping with such "openness." Even though, in principle, we employ mathematical modelling every time mathematics is applied outside its own domain, the pragmatic "Matematrix vocabulary" uses the terms model and modelling in relation to situations where an evident simplification of the modelled situation takes place, implying decisions, assumptions, the collection of information and data, etc. Hence, constructive mathematical modelling competency also involves elements that are not purely or even primarily of a mathematical nature, for example, knowledge of non-mathematical facts and considerations, as well as decisions regarding the model's purpose, suitability, relevance to the initiating questions, etc. (Niss & Jensen, 2002).

### *5.2. Textbook Approach*

In the Matematrix textbooks, tasks are categorized and presented with a view to lesson planning. When intending to challenge students' constructive mathematical modelling competency, this entails consistently and explicitly distinguishing between three types of tasks, which I describe and exemplify below: tasks inviting inquiry-based modelling, tasks inviting desk-based modelling, and Fermi tasks.

### *5.3. Tasks Inviting Inquiry-Based Modelling Projects*

The tasks in this category are developed and tested as starting points for modeling-oriented project work, where students work on the same topic for an extended period using inquiry-based approaches (Artigue & Blomhøj, 2013). Generally, and increasingly as students grow older and become more mature, such an approach involves out-of-classroom activities such as counting or measuring something at the school and/or communicating with other people at the school or in the local community. From our experience within the Matematrix group, the youngest students can only handle projects lasting 2-4 lessons, while secondary students can benefit from more prolonged periods of project work over 2-4 weeks.

Each book contains eight such tasks inviting so-called *inquiries* in a special section at the back of each book, leaving it up to the teacher to decide when to incorporate such periods of project work into the yearly lesson plan (Højgaard, 2025b). Following the same logic, the inquiries are not tied to specific chapters of the book and are often not linked to specific mathematical subject matter, as part of an open-ended approach. Instead, they are designed to support teachers in guiding students to develop either mathematical modelling competency (six in each book) or mathematical reasoning competency (two in each book).

Structurally, this implies that as students grow older and more mature, progression primarily concerns the *degree of coverage* in these inquiries, that is, "the extent to which all the aspects that define and characterize the competency form part of that individual's possession of the competency" (Niss & Højgaard, 2019). As the approach chosen to meet this ambition, each inquiry revolves around a single, very open-ended task or question. This task or question, accompanied by a short introductory text, establishes the topic of the inquiry and suggests a possible direction for the modeling (or reasoning) process. The rest of the two-page spread is used to provide inspiration for students in the form of pictures, diagrams, and either words and phrases they might use in online searches (in the books for older students) or more specific sub-questions (in the books for younger students, where more guidance is needed).

The following are examples from each grade (see Table 3 for more examples).

- Compare items found in the woods. (Matematrix 0, pp. 66-67).
- Draw a sketch of a room where you live. (Matematrix 1B, pp. 84-85).
- How many of you need to stand on top of each other to reach the ceiling? (Matematrix 2A, p. 85).
- How many students can your school accommodate? (Matematrix 3B, pp. 80-81).
- How many books are there in the school library? (Matematrix 4, pp. 150-151).
- Draw a map. (Matematrix 5, pp. 154-155).

- How much waste do you produce? (Matematrix 6, pp. 150-151).
- What do you use water for and how much do you use? (Matematrix 7, pp. 166-167).
- How many windmills should Denmark have? (Matematrix 8, pp. 172-173).
- Which means of transport is the best? (Matematrix 9, pp. 178-179).

These tasks have the potential to encourage students to work through every phase of the mathematical modelling process (cf. Figure 1) and thereby develop comprehensive mathematical modelling competencies. I have observed this potential firsthand when initiating and witnessing the implementation of these tasks in Danish classrooms as part of various research and development projects (Højgaard, 2019, 2021; Højgaard, 2025b).

When attempting to develop new tasks of this type, classroom trials can be supplemented or replaced by imagining how different groups of students might approach a given inquiry-based task if given the necessary support and guidance by their teacher and asking oneself questions such as: Does this task offer a natural jumping-off point for a constructive modelling process like the one depicted in Figure 3? And not least: How might students approach the initial motivation and systematisation phases of the modelling process, which lie at the core of constructive mathematical modelling competency, when working with this task? Højgaard (2021) provides an example of how this can be done, based on the final task listed above concerning different means of transport.

### *5.4. Tasks Inviting Desk-Based Modelling of Short Duration*

The tasks in this category are developed and tested to ensure they can be addressed within a relatively short time span. From a planning perspective, it is important that they can function within the framework of a single lesson. From an organizational perspective, the tasks are designed to be "desk-based"; they do not require out-of-classroom activities or the preparation of special materials by the teacher.

Such tasks are included alongside a range of other tasks in a section of each chapter of the Matematrix textbooks. As described above and exemplified in Table 2, the teachers' manual states which 3-4 competencies are in focus in each chapter a design principle where each of these competencies is accompanied by a list of tasks explicitly related to that specific competency. This often includes tasks related to constructive mathematical modelling competency, as this was one of the priorities when designing the books. As an example (cf. Table 3 for more examples), the following chapters and tasks in *Matematrix 5* have such a focus (cf. Matematrix 5, teachers' manual, pp. 12-14):

- Division: How long will it take you to save up DKK 1000? (Matematrix 5, p. 87).
- Scale: How much of the total area of the globe does Denmark occupy? (Matematrix 5, p. 126).
- Chance and experiments: How difficult is it to hit the bin with a paper ball? (Matematrix 5, p. 138).

As opposed to the inquiry-based tasks described above, tasks in this category focus on progression with respect to the *technical level* of constructive mathematical modelling competency that is, "the level and degree of sophistication of the mathematical concepts, results, theories, and methods which the individual can bring to bear when exercising the competency"(Niss & Højgaard, 2019).

### *5.5. Separate Pages with Fermi Tasks*

Some of the tasks involving desk-based modeling of short duration are called *Fermi tasks*, named after the Italian Nobel Prize-winning physicist (Enrico Fermi, 1954). Fermi tasks "are open, non-standard problems requiring the students to make assumptions about the problem situation and estimate relevant quantities before engaging in, often, simple calculations" (Ärlebäck, 2009, pp. 331-332).

In Fermi's field of physics, relevant quantities are described by specifying different variables, and assumptions about the problem situation are expressed through formulas relating these variables. We adopted this approach in the design of the Matematrix textbooks, which require students to have been introduced to and practiced working with formulas, particularly constructing their own formulas, before engaging with Fermi tasks. In Matematrix, such an introduction is provided in a chapter in Matematrix 6 (see Matematrix 6, pp. 108-109, 116). Following this introduction, Fermi tasks are presented on separate pages in selected chapters of Matematrix 6–9. Here is the complete list, including a translation of one of the tasks from each page.

- *Reality and mathematics:* How many students in your class have their birthday during the summer holidays? (Matematrix 6, p. 141).
- *Variables*: How many hours will you spend on mathematics during your life? (Matematrix 7, p. 20).
- *Algebra*: How much money will you make during your life? (Matematrix 8, p. 31).
- *Calculations of area*: How much of the Earth's area is mountainous? (Matematrix 8, p. 61).
- *Mathematical modelling*: How much of your life will you spend in front of a screen? (Matematrix 9, p. 20).
- *Volume and dimensions*: How large must a box be to contain 1000 matches? (Matematrix 9, p. 41).

The approach and layout of these pages serve three purposes. Firstly, we use Fermi tasks to introduce the mathematical modelling process, as suggested by Ärlebäck (2009), and to give students experience of the process without the need for a metacognitive overview. This is done by providing the same explicit point-by-point description of a modelling process at the top of each page with Fermi tasks (e.g., Jensen, Larsen, Pedersen, & Thorbjørnsen, 2014, p. 20).

"You can answer each question by

- Choosing some variables that you believe the answer depends on.
- Constructing a formula that demonstrates how to calculate using these variables.
- Making a qualified guess about the value of each variable.
- Inserting these values into the formula and calculating an approximate answer to the question.
- Assessing the approximate answer: Does it seem reasonable in relation to the question, or should the calculations be performed differently?

Secondly, the thematic page layout allows students to progress along a new dimension in their development of constructive mathematical modelling competency. Above, I have addressed how different types of tasks in the Matematrix books focus on two dimensions in this regard: degree of coverage and technical level. The pages with Fermi tasks focus on the third dimension of competency development suggested as part of the KOM framework (Niss & Jensen, 2002): the *radius of action*, representing "the range and variety of different contexts and situations in which the individual can successfully activate the competency" (Niss & Højgaard, 2019, p. 21). Dedicating entire pages to Fermi tasks provides an opportunity to use the same modeling approach, as specified at the top of the page, and the same conceptual focus, as outlined for that specific chapter, but with the various tasks spanning different contexts. According to the KOM framework's three-dimensional understanding of competency development, this means that the degree of coverage and the technical level are the same for all the tasks on each page, allowing students to focus on expanding their radius of action.

Thirdly, by presenting Fermi tasks with a fixed conceptual focus, we use mathematical modelling as a means of developing students' understanding of the concepts of variables and formulas, and the related symbols and formalism competency. In doing so, we follow *the promoting mathematics learning argument* for working with mathematical modelling in mathematics education (Blum & Niss, 1991), similar to the work with model-eliciting activities described above.

## 6. APPLICATION-REFLECTIVE COMPETENCY

### 6.1. Characterizing the Core Features of the Sub-Competency

Similar to what Greer and Verschaffel (2007) classify as critical modelling, I use the term *application-reflective competency* to describe the receptive facet of mathematical modelling competency. As such, this competency concerns an individual's insightful readiness to critically analyze mathematical models produced by others, cf. the conceptualization of mathematical modelling competency above.

Inevitably, this entails a working process based on mathematical modelling that has been performed by others. In rare cases, a mathematical model and the process through which it was developed might be presented and thus available for critique. More often, however, it is the *results* produced by a mathematical model that are presented, rather than the model itself. In such situations, the crux of the challenge is to recognise something as the result of mathematical modelling and, on this basis, exercise critique by approaching the modelling process in reverse, exploring the relationship between the result and the choices, mathematisations, and calculations required to achieve it. In the depiction of the mathematical modelling process in Figure 3, this corresponds to using knowledge of the characteristics of the phases labelled *motivation*, *systematisation*, *mathematisation*, and *mathematical analysis* when focusing on the phases labelled *interpretation* and *validation*. As an instance of *the critical competence argument* for working with mathematical modelling in mathematics education (Blum & Niss, 1991), Skovsmose (1990b) refers to this as exercising and developing *reflective knowledge* related to mathematical modelling.

### 6.2. Textbook Approach

How can and should such an ambition be pursued in mathematics education and facilitated by various materials? For general education in Denmark, which represents the target audience for the Matematrix books, a key perspective on that question stems from the purpose of mathematics education outlined in the national curriculum. Since 1995 (Undervisningsministeriet, 1995), this has been partly related to a goal of preparing students for democratic citizenship. In the current mathematics curriculum, this is framed as helping students "[…] appraise how mathematics is applied in order to take responsibility and exert an influence in a democratic society" (Børne- og Undervisningsministeriet, 2019, p. 3, my translation). As a source of inspiration for this ambition, Skovsmose (1990a) nicely sums up how it might be approached, based on two types of arguments:

Is it possible to develop the content and form of mathematical education in such a way that it may serve as a tool for democratization in both school and society? This question relates to two different arguments. The social argument of democratization states: (1) Mathematics has an extensive range of applications, (2) because of its applications, mathematics has a "society-shaping" function, and (3) in order to fulfill democratic obligations and rights, it is necessary to be able to identify the main principles of societal development. The pedagogical argument of democratization states: (1) Mathematical education has a "hidden curriculum," (2) the "hidden curriculum" of mathematical education in traditional forms implants a servile attitude towards technological questions in many students, and (3) we cannot expect the development of democratic competence in schools unless the teaching-learning situation is based on dialogue and the curriculum is not entirely determined from outside the classroom.

The social argument implies that we must aim at "empowering material," which could constitute a basis for reflective knowledge, i.e., knowledge about how to evaluate and criticize a mathematical model, while the pedagogical argument implies that we must aim at "open material," leaving space for decisions to be taken in the classroom.

In the Matematrix textbooks, we have addressed this challenge by providing two types of tasks designed to enhance students' application and reflective competencies. Both tasks encourage an open, inquiry-based approach: inquiry-based project work and application-critical thematic pages.

### *6.3. Tasks Inviting Inquiry-Based Application-Critical Projects*

The first type of task is a subset of the modelling-oriented inquiries described above, with the potential to challenge and develop the application-critical facet of mathematical modelling competency. This potential can be realized when inquiries revolve around an open-ended task or question that has been mathematically modelled by others, and when such models are available and accessible to students. In such cases, the teacher – or the students themselves, depending on age and level of maturity can choose to approach the inquiry from the perspective of application-reflective competency as an alternative or supplement to constructive mathematical modelling competency.

The following are examples of such tasks, deliberately including some examples also related to constructive mathematical modeling competency:

- How many students can your school accommodate? (Matematrix 3B, pp. 80-81).
- How much does it cost to have pets? (Matematrix 5, pp. 144-145).
- Draw a map. (Matematrix 5, pp. 154-155).
- How much waste do you produce? (Matematrix 6, pp. 150-151).
- What do you use water for and how much do you use? (Matematrix 7, pp. 166-167).
- How does one determine the price of a product? (Matematrix 8, pp. 176-177).
- How much do I cost? (Matematrix 9, pp. 176-177).

### *6.4. Application-Critical Thematic Pages*

A second type of task in the Matematrix books more directly and explicitly seeks to challenge and develop students' application-reflective competency. In each task, students are given a snapshot of a news story and a link to the full text. They are then asked to critically reflect on how the media uses mathematics in their coverage of the story. Like the Fermi tasks, these tasks are provided on separate pages in selected chapters, but only in Matematrix 7-9, as the Matematrix group believed that such tasks require students to have reached a certain age and level of maturity. Here is the complete list, including a translation of one of the tasks from each page.

- *Differences*: "We are European champions in waste." (Matematrix 7, pp. 144-145).
- *Probability*: "A 1.6-ton part of a satellite hits the Earth in one piece." (Matematrix 8, p. 99).
- *Mathematical modelling*: "Danish kids 'cost' DKK 1 million on average." (Matematrix 9, p. 21).
- *Statistics*: "Shops let young people gamble illegally." (Matematrix 9, pp. 62-63).
- *Functions and graphs*: "Michael has earned DKK 117,000 from his Vestas stocks." (Matematrix 9, p. 78).

As can be seen, these thematic pages have a similar form to the pages with Fermi tasks. The same explicit point-by-point approach is outlined at the top of each page, this time with an application-critical focus (e.g., Jensen et al., 2014, p. 144):

- "What is the text about?
- Where do you believe mathematics has been applied?
- In which ways has mathematics been applied? Illustrate with drawings and calculations.
- Is this a good and fair way to apply mathematics?"

For the same reasons as for the Fermi pages, these pages can be used to develop students' application-reflective competency focusing on the *radius of action* dimension. Dedicating entire pages to the same type of task provides an opportunity for students to use the same application-critical approach, as specified at the top of the page, and the same conceptual focus, as specified for that specific chapter, but with the various tasks spanning different contexts.

## 7. MATHEMATISATION COMPETENCY

### *7.1. Characterizing the Core Features of the Sub-Competency*

I use the term *mathematisation competency* to describe an individual's insightful readiness to solve problems defined as such by a challenge to mathematise. Loosely speaking, mathematisation competency combines mathematical problem-solving skills and mathematisation. These two "ingredients" are elaborated in Højgaard (2021). Here, I will provide a brief summary.

I use *problem* to denote a situation involving a number of methodologically open questions that pose an intellectual challenge for someone without direct access to methods, procedures, or algorithms that would enable them to answer these questions (Blum & Niss, 1991). *Problem solving* is used, plain and simple, to denote the process of trying to solve a problem. Crucial to this process is that it is characterized by the need for conscious or unconscious reflection on method. *Mathematical problem solving* is when a defining feature of this reflection process is that it involves mathematical terms, methods, and results.

From the conceptual point of view described above, conducting a mathematical modelling process will often entail solving one or more mathematical problems, not least during the mathematisation and mathematical analysis phases (phases (c) and (d) in Figure 3). However, there are key differences at the core of the two sub-competencies. As elaborated above, the crux of the challenge of constructive mathematical modelling competency is to learn how to deal with the many choices that need to be made before mathematical concepts and techniques can be applied. In contrast, due to its focus on methodological reflections, mathematical problem solving concerns the ability to cope with what can be characterized as a feeling of "knowing what the goal is without knowing how to achieve it"(Blomhøj & Jensen, 2003). I have argued that the ability to cope with what can be a quite frustrating sense of being cognitively stuck is a core feature of mathematical problem-solving competence (Jensen, 2007).

### *7.2. Textbook Approach*

The following are examples of tasks from selected Matematrix chapters (see Table 1 for more examples):

- How long is the removal van when the boot is open? (*Choosing mode of calculation*: Matematrix 2B, pp. 35-36).
- How many migratory journeys must a lapwing make to equal the distance flown by an Arctic tern? (*Choosing mode of calculation*: Matematrix 3B, pp. 66-67).
- There are 20 students in 4C at Brøkvild School. Of these, nine are boys. Write the share of female students as a fraction. (*Fractions*: Matematrix 4, p. 47).
- Johan's father is 7 years older than Johan's mother. Their combined age is 83. How old are each of Johan's parents? (*Equations*: Matematrix 6, pp. 76-77).
- Which is the better option for shops: To add a percentage profit before or after VAT? Justify your answer. (*Percent*: Matematrix 8, p. 79).

These are examples of a type of task that only challenges students to work with phases (c), (d), and (e) of the mathematical modeling process as depicted in Figure 3. The delimitation of the context and task in phases (a) and (b) is already addressed in the task formulation, and the inclination to work with phase (f) often results from having worked with phases (a) and (b) (Christiansen, 2001).

Tasks like these, which I consider to be similar to the model-eliciting activities mentioned above, challenge the students' competency to mathematise a more or less well-defined non-mathematical problem. Hence, following the conceptualization above, such tasks should mainly be seen as an invitation to develop problem-solving competency within the domain of applied mathematics. They do not challenge and therefore cannot be used to develop-mathematical modelling competency with a full degree of coverage, but if properly formulated and orchestrated, such tasks can challenge what is often a vital part of developing this competency: applied mathematical problem solving focused on mathematization.

## 8. CONCLUSION

In the introduction, I posed the following research question.

How can textbooks be designed to systematically facilitate students' development of mathematical modelling competency, and what types of tasks best support this development?

From an exemplary perspective, I have described how my colleagues and I addressed this question when developing and writing the Matematrix series of mathematics textbooks for grades K-9. Based on this experience, my response to the research question can be summarized as follows:

Students' development of mathematical modelling competency can be facilitated using a two-dimensional content model to systematically and explicitly address three sub-competencies:

- *Constructive mathematical modelling competency* by means of a) tasks inviting inquiry-based modelling projects, b) tasks inviting desk-based modelling of short duration, and c) separate pages with Fermi tasks.
- *Application-reflective competency* by means of a) tasks inviting inquiry-based application-critical projects, and b) application-critical thematic pages.
- *Mathematisation competency* by means of tasks facilitating mathematisation across the entire range of conceptual foci.

Exemplary task types addressing each of these competencies are summarized in Table 3.

**Table 3**. Examples of tasks from the textbook series Matematrix, developed for and used in primary and lower secondary education with the explicit aim of facilitating the development of mathematical modelling competency.

| **Educational level**<br>**Tasks inviting students' development of...** | | ***Grades K-3*** | ***Grades 4-6*** | ***Grades 7-9*** |
|---|---|---|---|---|
| Constructive mathematical modelling competency | – *Inquiry-based projects (2-4 lessons or weeks)* | • Compare things found in the woods.<br>• Create your own shop and visit your classmates' shops.<br>• How many of you need to stand on top of each other to reach the ceiling?<br>• How big is the school playground?<br>• How many students can your school accommodate? | • How many books are there in the school library?<br>• How do you spend your time?<br>• How much does it cost to have pets?<br>• Draw a map.<br>• What does a litre look like?<br>• How much waste do you produce? | • How much water do you use?<br>• How much do I cost?<br>• How can one navigate?<br>• How many windmills should Denmark have?<br>• What is the relation between one's income and the tax one pays?<br>• Which means of transport is the best? |
| | – *Desk-based work of short duration (within a single lesson)* | • How high can you jump?<br>• Who are tallest: the girls or the boys?<br>• Draw a sketch of one of the rooms in your house.<br>• How long does it take to read a book?<br>• How long does it take to count to 1000?<br>• How many students weigh the same as an elephant? | • How long will it take you to save up DKK 1000?<br>• How much of the total area of the globe does Denmark fill?<br>• What is the distance between your ears?<br>• How big is your arm?<br>• Fermi task: How many students in your class have their birthday during the summer holidays? | • Draw a floor plan of a 135 m² house.<br>• Draw a graph showing how the temperature of a glass of water changes when you add ice cubes.<br>• Fermi task: How much money will you make during your life?<br>• Fermi task: How much of your life will you be spending in front of a screen? |
| Mathematical application-critical competency | – *Inquiry-based projects (2-4 lessons or weeks)* | • How big is the school playground?<br>• How many students can your school accommodate? | • How do you spend your time?<br>• How much does it cost to have pets?<br>• Draw a map.<br>• How much waste do you produce? | • How much water do you use?<br>• How does one determine the price of a product?<br>• How much do I cost?<br>• How can one navigate?<br>• What is the relation between one's income and the tax one pays? |
| | – *Desk-based work of short duration (within a single lesson)* | | | • "We are European champions in waste."<br>• "A 1.6-ton part of a satellite hits the Earth in one piece."<br>• "Danish kids "cost" DKK 1 million on average." |
| Mathematisation competency | – *Desk-based work of short duration (within a single lesson)* | • How many more girls are there than boys in the drawing? | • There are 20 students in 4C at Brøkvild School. Of these, nine are | • Three children are to share DKK 450. Mark gets DKK 50 more than Eva, and |

| | | | |
|---|---|---|---|
| | • How long is the removal van when the boot is open?<br>• What fraction of the balloons are either blue, red or yellow?<br>• How many migratory journeys must a lapwing make to equal the distance flown by an Arctic tern? | boys. Write the share of female students as a fraction.<br>• Johan's father is 7 years older than Johan's mother. Their combined age is 83. How old are each of Johan's parents? | Patricia gets twice as much as Eva. How much do they each get?<br>• What is the better option for shops: To add a percentage profit before or after VAT? Justify your answer. |

**Source:** Adapted from Højgaard (2021).

## 9. DISCUSSION

Is the research presented in this article relevant? If so, for whom? There are at least two aspects of that question worth discussing: its relevance for mathematics education practice and for research. Let me address them one at a time.

When it comes to *the practice of mathematics education*, I believe the relevance of this research lies at its very roots: The research question was posed because of (and hence framed by) developmental needs related to the practice of mathematics education, and the answers to this question have been fully implemented in that practice. In more concrete terms: my colleagues and I agreed there was a need to develop a way of facilitating the teaching of mathematical modelling in Danish K-9 classrooms, and, as luck would have it, we were offered the opportunity to develop a new series of textbooks – Matematrix – with this need as one of the driving forces. However, there was a need for research studying how this might be approached within the competency-focused KOM framework. At the time, I was a research student in the field of mathematics education with a particular interest in the teaching of mathematical modelling (Jensen, 2007).

Hence, it was only natural that I took the lead in this endeavor, both in conducting the necessary research and applying the findings when developing the textbooks.

This article attempts to report on both research and its application by providing a general framework for facilitating the development of mathematical modelling competency through a series of textbooks, along with numerous concrete examples of tasks designed and developed in accordance with this framework. Some might argue that this does not meet the criteria of a research article; however, I believe there is a need for more research that systematically attempts to bridge the gap between research and practice in mathematics education.

When it comes to *the relevance of this article for research*, it shares the fundamental challenge of *generality* with any other case study: the knowledge developed in the study is relevant for those involved, but who else might benefit? Aside from the members of the Matematrix group, who else can use knowledge of how we approached the facilitation of teaching focused on developing students' mathematical modelling competency?

I hope that other researchers who share this developmental ambition find the article relevant. Not necessarily in the sense of being able to directly translate the chosen textbook design to a different context, but with a research interest in both task design and mathematical modelling competency. However, in accordance with the naturalistic paradigm, I acknowledge that "at best only working hypotheses may be abstracted, the *transferability* of which is an empirical matter, depending on the degree of similarity between sending and receiving contexts" (Lincoln & Guba, 1985, p. 297, italics in original).

That is, I use my experiences from participating in the development of the Matematrix books to analyze and describe what I have found to be promising approaches when attempting to facilitate the teaching of mathematical modeling competency. It is for the readers of this article to decide the extent of its relevance in other circumstances.

With this in mind, I believe that there are two reasons why it is relevant to consider the development and design of the Matematrix textbooks *a critical case*, "defined as having strategic importance in relation to the general problem" (Flyvbjerg, 2006, p. 229).

Firstly, Denmark can be considered something of a best-case scenario for exploring competency-oriented teaching because many circumstances have aligned to make such an approach both possible and desirable, cf. the introduction to this article and the description and analysis of the development of competency-oriented mathematics curricula in Denmark in Højgaard and Sølberg (2023).

Secondly, as mentioned in the introduction, one of the main ambitions behind the didactic design of the Matematrix books was an attempt to systematically facilitate the incorporation of mathematical competencies as a key element in mathematics education for grades K-9 in Denmark. As one of the national experts in such an endeavor, I was a natural and enthusiastic choice to be the driving force in this process, with my efforts met with resounding support from the books' editors and co-authors. Hence, the Matematrix series is arguably a best-case attempt to use textbooks to facilitate competency-oriented mathematics teaching in general and the teaching of mathematical modelling competency in particular, and, as such, is critical for exploring this endeavor.

Of course, they do not provide step-by-step instructions for others to follow; no such thing exists when it comes to mathematics education.

Rather, the Matematrix textbook series is a critical case for textbook-based facilitation of students' development of mathematical modelling competency in the sense of being grounded in a very thorough and research-based analytical approach and written under very favorable conditions, both locally and nationally.

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